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Note

Direct Estimate for Bernstein Polynomials

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For the Bernstein polynomials

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \tag{1}$$

the pointwise approximation

$$|B_n(f,x) - f(x)| \le C\omega_{\varphi^\lambda}^2\left(f, n^{-1/2}\varphi(x)^{1-\lambda}\right),$$
$$0 \le \lambda \le 1, \qquad \varphi(x)^2 = x(1-x) \tag{2}$$

will be proved. This estimate yields a treatment that unifies the classical estimate for $\lambda = 0$ (see [ST]) and the norm estimate for $\lambda = 1$ (see [DT, p. 117]). Moreover, (2) yields an analogue to the pointwise algebraic polynomial approximation in $C[-1,1]$ which was proved recently (see [DJ]).

We recall that

$$\omega_{\varphi^\lambda}^2(f,t) = \sup_{0<h\le t} \sup_{x \pm h\varphi^\lambda(x) \in [0,1]} |f(x - h\varphi^\lambda(x) - 2f(x)$$
$$+ f(x + h\varphi^\lambda(x))| \tag{3}$$

is equivalent to the K-functional

$$K_{\varphi^\lambda}(f,t^2) = \inf\left(\|f-g\|_{C[0,1]} + t^2\|\varphi^{2\lambda}g''\|_{C[0,1]}\right) \tag{4}$$

(where the infimum is taken on functions satisfying $g, g' \in A, C_{\text{loc}}$). That is

$$C^{-1}K_{\varphi^\lambda}(f,t^2) \le \omega_{\varphi^\lambda}^2(f,t) \le CK_{\varphi^\lambda}(f,t^2) \tag{5}$$

(see [DT], Chap. 2]).

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Proof of the Estimate (2). Using (4) and (5), we may choose $g_n \equiv g_{n,x,\lambda}$ for a fixed x and λ such that

$$\|f - g_n\|_{C[0,1]} \le A\omega^2_{\varphi^\lambda}\big(f, n^{-1/2}\varphi(x)^{1-\lambda}\big) \tag{6}$$

and

$$n^{-1}\varphi(x)^{2-2\lambda}\|\varphi^{2\lambda}g''_n\| \le B\omega^2_{\varphi^\lambda}\big(f, n^{-1/2}\varphi(x)^{1-\lambda}\big). \tag{7}$$

Hence we have

$$\begin{aligned}
&|B_n(f,x) - f(x)| \\
&\quad\le |B_n(f - g_n, x) - (f(x) - g_n(x))| + |B_n(g_n, x) - g_n(x)| \\
&\quad\le 2A\omega^2_{\varphi^\lambda}\big(f, n^{-1/2}\varphi(x)^{1-\lambda}\big) + |B_n(g_n, x) - g_n(x)|.
\end{aligned}$$

We now write, following [DT, p. 141],

$$\begin{aligned}
&|B_n(g_n, x) - g_n(x)| \\
&\quad\le \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\left|\int_{k/n}^{x}(xv)g''_n(v)\,dv\right| \\
&\quad\le \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\frac{|x - k/n|}{\varphi(x)^{2\lambda}}\left|\int_{k/n}^{x}\varphi(v)^{2\lambda}|g''_n(v)|dv\right| \\
&\quad\le \|\varphi^{2\lambda}g''_n\|\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\frac{(x - k/n)^2}{\varphi(x)^{2\lambda}} \\
&\quad\le \|\varphi^{2\lambda}g''_n\|n^{-1}\varphi(x)^{2-2\lambda}
\end{aligned}$$

which completes the proof of (2).

Remark. Similar estimates for other linear operators follow the same proof.

References

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